Def. 1 and Thm. 2 center around the sets $S^I \subset J^k(M,N)$, defined so that for a good map $f \in C^\infty(M,N)$ we have $\Sigma^I(f) = (j^k f)^{-1}(S^I)$.

Their definition is based on the

<u>intrinsic derivative</u>:

<u>4. Def</u>

Let E, F be two vector bundles over a mfd M and $\rho: E \to F$ a bundle map

$$\left(\begin{tikzcd} E \arrow[rr, "\rho"] \arrow[dr, "\pi_M"'] & \circlearrowleft & F \arrow[dl, "\pi_F"] \\ & M & \end{tikzcd} \iff \rho(\underset{\pi_E^{-1}(x)}{\underset{\shortparallel}{E_x}}) \subset \underset{\pi_F^{-1}(x)}{\underset{\shortparallel}{F_x}} \right)$$

Locally $E = M \times \mathbb{R}^k$, $F = M \times \mathbb{R}^\ell$
and $\rho: M \to Hom(\mathbb{R}^k, \mathbb{R}^\ell)$

so $d\rho(x): T_x M \to T_{\rho(x)} Hom(\mathbb{R}^k, \mathbb{R}^\ell) = Hom(\mathbb{R}^k, \mathbb{R}^\ell)$

For $K_x := \ker \rho(x)$, $L_x := \operatorname{coker} \rho(x) = \mathbb{R}^\ell / \operatorname{im} \rho(x)$

the intrinsic derivative $D\rho(x)$ of ρ at x is defined as

$$D\rho(x) : T_x M \xrightarrow{d\rho(x)} \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^\ell) \xrightarrow[\substack{\uparrow \\ \text{restrict \& project}}]{} \mathrm{Hom}(K_x, L_x),$$

- independent of coord. on M ✓
- " of trivialization:

Let ϕ, ψ be bundle isom. on E and F

i.e. $\phi : M \to \mathrm{Aut}(\mathbb{R}^k)$, $\psi : M \to \mathrm{Aut}(\mathbb{R}^\ell)$

ρ transforms to $\tilde{\rho}$, $\tilde{\rho}(x) = \psi(x)\, \rho(x)\, \phi^{-1}(x)$

and ϕ and ψ induce isos

$$K_x \cong \tilde{K}_x = \ker \tilde{\rho}(x) \quad \text{and} \quad L_x \cong \tilde{L}_x = \operatorname{coker} \tilde{\rho}(x)$$

so get a map

$$H_{\phi\psi} : \mathrm{Hom}(K_x, L_x) \longrightarrow \mathrm{Hom}(\tilde{K}_x, \tilde{L}_x)$$
$$A \longmapsto \psi(x)\, A\, \phi^{-1}(x)$$

To show:

$$\begin{tikzcd} T_x M \arrow[r, "D\rho(x)"] \arrow[dr, "D\tilde{\rho}(x)"'] & \mathrm{Hom}(K_x, L_x) \arrow[d, "H_{\phi\psi}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Hom}(\tilde{K}_x, \tilde{L}_x) \end{tikzcd}$$

1. If Φ, Ψ linear, i.e. $\Phi = id_M \times \varphi$, $\Psi = id_M \times \psi$

then $d\tilde{\rho}(x) = \psi \, d\rho(x) \, \varphi^{-1}$

2. use 1. to get $\Phi(x) = id_{\mathbb{R}^k}$, $\Psi(x) = id_{\mathbb{R}^l}$

$\Rightarrow K_x = \tilde{K}_x$, $L_x = \tilde{L}_x$

and $D\tilde{\rho}(x) = D\rho(x)$ iff $d\tilde{\rho}(x) = d\rho(x)$

$$d\tilde{\rho}(x) = d\left(\Psi \circ \rho \circ \Phi^{-1}\right)(x)$$

$$= d\Psi(x)\, \rho(x)\, \Phi^{-1}(x) + \Psi(x)\, d\rho(x)\, \Phi^{-1}(x) + \Psi(x)\, \rho(x)\, d\Phi^{-1}(x)$$

$$= \underbrace{d\Psi(x)\, \rho(x)}_{\substack{=0 \\ \text{on } \ker \rho(x)}} + d\rho(x) + \underbrace{\rho(x)\, d\Phi^{-1}(x)}_{\in \operatorname{im} \rho(x)} .$$

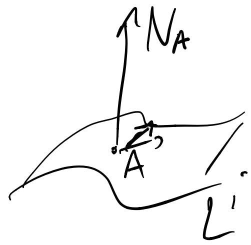
✓

There is an invariant definition of D which uses the following fact:

Recall Lemma II.9: The set of corank i maps $L^i(V,W)$ is a submf of $\operatorname{Hom}(V,W)$ (V,W vect. spaces)

For $A \in L^i(V,W)$ set $K_A = \ker A$, $L_A = \operatorname{coker} A$ and denote by N_A the <u>normal space</u> at A,

$$N_A := T_A \operatorname{Hom}(V,W) \big/ T_A L^i(V,W)$$



<u>5. Lemma</u> $\quad N_A \cong \operatorname{Hom}(K_A, L_A)$

<u>Proof:</u> (cf. Lemma II.9)

For $RP : T_A \operatorname{Hom}(V,W) \cong \operatorname{Hom}(V,W) \longrightarrow \operatorname{Hom}(K_A, L_A)$

$\ker(RP) = T_A L^i(V,W)$ because

wlog $A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ $\quad r = \operatorname{rank} A$

locally $L^i(V,W) = F^{-1}(0)$ where

$$F : \begin{pmatrix} S & T \\ U & V \end{pmatrix} \mapsto V - US^{-1}T$$

and $dF(A) = RP \Rightarrow \underset{\underset{\ker RP}{\shortparallel}}{\ker dF(A)} = T_A L^i(V,W)$

and thus

$$N_A = T_A \operatorname{Hom}(V,W) \big/ T_A L^i(V,W) \cong \operatorname{Hom}(K_A, L_A) \qquad \blacksquare$$

Now let E, F be vect. bundles over M,

$\sigma \in L^i(E_x, F_x) = L^i(E,F)_x$ $\left(\begin{array}{c} L^i(E,F) \subset \mathrm{Hom}(E,F) \\ \pi \downarrow \\ M \end{array} \right)$

Lemma 5 "lifts" to the category of vector bundles,

normal space to $L^i(E,F)$ in $\mathrm{Hom}(E,F)$ at σ $\rightsquigarrow N_\sigma \cong \mathrm{Hom}(K_\sigma, L_\sigma)$.

For $\rho: E \to F$ bundle map, view it as a map $\rho: M \to \mathrm{Hom}(E,F)$ and let $\sigma = \rho(x) \in L^i(E,F)$. Then we get a map

restr. & proj.

$$D\rho(x): T_x M \xrightarrow{d\rho(x)} T_\sigma \mathrm{Hom}(E,F) \longrightarrow N_\sigma \cong \mathrm{Hom}(K_\sigma, L_\sigma)$$

<u>6. Prop</u>: Supp $\rho(x) = \sigma \in L^i(E,F)$. Then the following are equivalent:

1. $D\rho(x): T_x M \longrightarrow \mathrm{Hom}(K_\sigma, L_\sigma)$ surj.
2. $\rho \pitchfork L^i(E,F)$ at x.

e.g. (main example / cf. exercise)

$f: M \to N$ smooth. Set $E = TM$,

$F = f^* TN = \{ (x,v) \mid f(x) = \pi_F(v) \} \subset M \times TN$

and $\varphi = df$:

$\rightarrow$ $D(df)(x): T_x M \xrightarrow{\text{linear}} \mathrm{Hom}(\ker df(x), \operatorname{coker} df(x))$

$= T_x M \otimes \ker df(x) \to \operatorname{coker} df(x)$

$\rightarrow$ determined by $j^2 f$!

$\rightarrow$ bilinear symm map $d^2 f(x): \ker df(x)^2 \to \operatorname{coker} df(x)$

(if $N = \mathbb{R}$, get $d^2 f(x) = Hf(x)$ "Hessian").

Intrinsic derivatives allow to construct the $S^I \subset J^{|I|}(M,N)$.

Sketch for $|I| = 2$:

1. $J^1(M,N) \supset S^i \cong L^i(TM, TN)$ (corank i bundle maps)

For $\sigma \in S^i$, $s(\sigma) = x$, $t(\sigma) = y$, set

$K_\sigma = \ker \sigma \subset T_x M$ and $L_\sigma = \operatorname{coker} \sigma \subset T_y N$

$\to$ vector bundles $\begin{array}{c} K \\ \downarrow \\ S^i \end{array}$ and $\begin{array}{c} L \\ \downarrow \\ S^i \end{array}$

2. Lemma 5 ("Lift"): The normal bundle to S^i in $J^1(M,N)$ is isom. to $\mathrm{Hom}(K,L)$ (as bundles over M)

3. Define $S^{i(2)} := (\pi_1^2)^{-1}(S^i) \subset J^2(M,N)$.

Intrinsic derivative induces a surjective bundle map

$$\begin{tikzcd} S^{i(2)} \arrow[rr, "\Delta"] \arrow[dr] & & \mathrm{Sym\,Bil}(K \times K, L) \arrow[dl] \\ & S^i & \end{tikzcd}$$

$$\mathrm{Sym\,Bil}(K \times K, L) = \mathrm{Hom}(K \circ K, L)$$

$\uparrow$ symm. product

4. V, W vect. spaces. For $\pi: V \otimes V \to V \circ V$ with $\ker \pi = V \wedge V$ consider a map

$$I: \mathrm{Hom}(V \circ V, W) \to \mathrm{Hom}(V \otimes V, W) \to \mathrm{Hom}(V, \mathrm{Hom}(V,W))$$

$$B \mapsto B \circ \pi \quad , \quad A \mapsto (A(v): v' \mapsto A(v \otimes v'))$$

Define $\mathrm{Hom}(V \circ V, W)_j := I^{-1}(L^j(V, \mathrm{Hom}(V,W)))$

$\neq L^j(V \circ V, W)$!

Claim: $Hom(V \circ V, W)_j \subset Hom(V \circ V, W)$ is a submf of codim

$$c_j(v,w) = \frac{w}{2} v(v+1) - \frac{w}{2}(v-j)(v-j+1) - j(v-j)$$

(see G&G)

5. 4. lifts to vector bundles:

$Hom(E \circ E, F)_j$ is a subbundle of $Hom(E \circ E, F)$ of the same codim $c_j(\dim E_x, \dim F_x)$

6. Δ is submersion, so

$S^{ij} := \Delta^{-1}(Hom(K \circ K, L)_j) \subset S^{i(2)}$ is a submf of codim $c_j(\dim K, \dim L)$.

7. Thm: If f one-generic, then

$$x \in \Sigma^{ij}(f) \iff j^2 f(x) \in S^{ij}.$$

Proof:

Let $j^2 f(x) = \sigma \in S^i$. Normal space to S^i in $J^1(M,N)$ at σ is $N_\sigma \cong Hom(K_\sigma, L_\sigma)$

Prop 6: $(j^2 f \pitchfork S^i) \Rightarrow D(df)(x) : T_x M \to \mathrm{Hom}(K_\sigma, L_\sigma)$ is surjective and its kernel is $T_x \Sigma^i(f)$.

Suppose $x \in \Sigma^{ij}(f) = \Sigma^j(f|_{\Sigma^i(f)})$. Then

$$\underbrace{\ker D(df)(x)}_{= T_x \Sigma^i(f)} \cap \underbrace{K_\sigma}_{= \ker df(x)} \text{ is a } j\text{-dim subspace.}$$

Therefore, $D(df)(x)\big|_{K_\sigma}$ has j-dimensional kernel

$$\overset{\text{Exercise}}{\iff} j^2 f(x) \in \mathrm{Hom}(K_\sigma \circ K_\sigma, L_\sigma)_j$$

$$\iff j^2 f(x) \in S^{ij}.$$ ∎

8. Corollary

If $j^2 f \pitchfork S^{ij}$, then $\Sigma^{ij}(f) \subset \Sigma^i(f)$ is a submf of codim

$$C_j(\, i + \max(m-n, 0),\ i + \min(n-m, 0)\,).$$

Thm V.4: $\{ f \mid \forall i,j \ \ j^2 f \pitchfork S^{ij} \}$ is dense in $C^\infty(M,N)$. Such maps are called two-generic (or simply good).